Asymptotic Nature of First Order Neutral Delay Difference Equation

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Abstract

In this paper, the presence of non oscillatory solutions having asymptotic nature of the 1st order NDDE with variable coefficients and delays are contemplated. Some new adequate conditions are given. Specifically, conditions given in this paper are adynamic than those known, so the outcomes in this paper have more extensive application than the current ones.

Keywords

Nonoscillatory solution, asymptotic nature ,First order, Neutral, Delay difference equation, Variable coefficients.

1 Introduction

The investigation of the asymptotic and oscillatory conduct of the solutions of delay difference equations exhibits a solid hypothetical intrigue. Beside the mathematical intrigue, the investigation of those conditions is persuaded by their applications. Delay difference equations emerge in a few regions of connected science, counting circuit hypothesis, bifurcation examination, populace flow, delay difference equations are utilized as a part of the examination of PC systems containing lossless and in mobile communication[24].

In this article, we investigate on the following 1st order delay difference equation with variable coefficient.

(1) $\Delta(z_n) + p_n G\left(y_{\sigma(n)}\right) = f_n, n \geq n_0$,Where

(2) $z_n = y_n - p_n y_{\tau(n)}$, $< f_n >, < p_n >$ are real sequence, $< q_n >$ is a positive real sequence. $G(x)$ is a continuou function from R to R such that $xG(x) > 0$. $\sigma(n), \tau(n)$ are monotonic increasing function such that $\sigma(n) \leq n$, $\tau(n) \leq n$

Our result hold for $f_n = 0$. That is

(2*) $\Delta(z_n) + p_n G\left(y_{\sigma(n)}\right) = 0, n \geq n_0$

We assume the following conditions for the our work in the sequel.

(C0) $xG(x) > 0 \;\; for \;\; x \neq 0$

(C1) G is nondecreasing

(C2) $\left|\sum_{n=0}^{\infty} f_n\right| < \infty$

(C3) $\sum_{n=0}^{\infty} q_n = \infty$

(D1) $0 \leq p_n \leq p < 1$

(D2) $-1 < -p \leq p_n \leq 0$

(D3) $-d \leq p_n \leq -c < -1$

(D4) $1 \leq c \leq p_n < d$

Where p,c,d are positive real numbers

2 THE MAIN RESULTS

Recently there are many results are published concerning oscillatory and non oscillatory of first order differential equations. Some fundamental lemma's are presented here (see [8],[9],[23])

Lemma:1

Let y_n be a real sequence for $n \geq n_0$ such that y_n is negative for large n and $\tau(n) \leq n$ is an unbounded strictly increasing function .Then

(i) $y_n < y_{\tau(n)}$ for large n ,implies $\limsup_{n\to\infty} y_n \neq 0$

(ii) $y_n > y_{\tau(n)}$ for large n ,implies y_n is bounded.

Remark:1

Let y_n be a real sequence for $n \geq n_0$ such that y_n is positive for large n and $\tau(n) \leq n$ is an unbounded strictly increasing function .Then

(i) $y_n > y_{\tau(n)}$ for large n ,implies $\liminf_{n\to\infty} y_n \neq 0$

(ii) $y_n < y_{\tau(n)}$ for large n ,implies y_n is bounded.

Remark:2

Since $F_n = \sum_{i=n}^{\infty} f_i$,therefore $\lim_{n\to\infty} F_n = 0$ by (C2) .Hence $\lim_{n\to\infty} w_n = 0 = \lim_{n\to\infty} z_n = 0$

Lemma :2

If y_n is a negative real sequence for $n \geq \alpha$,such that it is bounded. Then there exists a subsequence of points $\{n_m\}$ such that $m \to \infty \Rightarrow \{n_m\} \to -\infty$ and $\{y_{n_m}\} \to -\infty$ and $\{y_{n_m}\} = \min\{y_n : \alpha \leq n \leq n_m\}$

Lemma: 3

Suppose that $\tau(n)$ is a continuous monotonically increasing unbounded function such that $\tau(n) \leq n$ .Let $\{u_n\},\{v_n\}$ be real sequence for $n \geq n_0$ such that $u_n = v_n - pv_{\tau(n)}$, $n \geq \tau_{-1}(n_0)$,where is the inverse function of τ , $p \in R$ *and* $p \neq -1$ .Assume that $\lim_{n\to\infty} u_n = l \in R$ *exists.* Then the following statements are holds good.

(i) $\liminf_{n\to\infty} v_n = a \in R$ then $l = (1-p)a$

(ii) $\limsup_{n\to\infty} v_n = b \in R$ then $l = (1-p)b$

Theorem: 2.2.41

Assume that y_n is an eventually negative solution of (1).Let (C0-C3) hold and set

(3) $w_n = z_n + F_n$ and

(4) $F_n = \sum_{i=n}^{\infty} f_i$

Then the following statement are true.

(a) w_n is an eventually increasing sequence.

(b) Suppose that $f_n \geq 0$ .If $p_n \geq 1$ then $w_n > 0$

(c) If $0 \leq p_n \leq 1$ then $w_n < 0$, $\limsup_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} w_n = 0$

Proof:

(a)

From (1) and (3) we obtain ,

(5) $$\Delta w_n = \Delta(z_n + F_n) = \Delta\left(y_n - p_n y_{\tau(n)}\right) = -q_n G\left(y_{\sigma(n)}\right) \geq 0$$

So Δw_n is eventually increasing sequence.

(b)

Suppose $p_n \geq 1$,Since $f_n \geq 0$ so $F_n \geq 0$ .To prove that $w_n > 0$ .If not let $w_n \leq 0$ .Then from (2.1a) it implies that $z_n \leq 0$ .That is $y_n - p_n y_{\tau(n)} \leq 0$ .This implies that $y_n \leq p_n y_{\tau(n)}$ .Since $p_n \geq 1$ and y_n is negative so $p_n y_{\tau(n)} \leq y_n$ .Hence $y_n \leq p_n y_{\tau(n)} \leq y_n$

By lemma (1(i)) , $\limsup_{n\to\infty} y_n \neq 0$. That is y_n is bounded above by a negative constant say m. That is $y_n <$ m for $n \geq n_0$ .This implies

$$y_{\tau(n)} < m \text{ for } n \geq n_1$$
$$\Rightarrow G\left(y_{\tau(n)}\right) < G(m)$$
$$\Rightarrow -q_n G\left(y_{\tau(n)}\right) \geq -G(m)$$

It is true since (C1) holds good.

From(5) $\Delta w_n \geq -q_n G\left(y_{\tau(n)}\right)$. Taking summation of both sides from N to n-1 ,we get ,

$$\sum_{i=N}^{n-1} \Delta w_n \geq -\sum_{i=N}^{n-1} q_n G\left(y_{\tau(n)}\right)$$

$$\Rightarrow w_n - w_N \geq -G(m)\sum_{i=N}^{n-1} q_i \to \infty \ as \ n \to \infty$$

So $\lim_{n\to\infty} w_n = \infty$,which contradicts the fact that w_n is non increasing sequence. for large n.So our assumption is wrong.

Hence w_n>0

(c)

Let $0 \leq p_n \leq 1$. Since w_n is increasing , $\lim_{n\to\infty} w_n = l$,where $-\infty < l \leq \infty$ .Then $\lim_{n\to\infty} F_n = 0 \Rightarrow \lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n$. If $l = \infty$ then $\lim_{n\to\infty} z_n = \infty$ $\Rightarrow z_n > 0$ for large n .So $y_n - p_n y_{\tau(n)} > 0 \Rightarrow y_n > p_n y_{\tau(n)} \geq y_{\tau(n)}$,since $p_n \leq 1$.

Therefore by remark (1) y_n is bounded and consequently w_n is bounded ,a contradiction to the fact that $l = \infty$.Thus $-\infty < l < \infty$. Now we claim that

(6) $\limsup_{n\to\infty} y_n = 0$ .Otherwise there exist N and α such that $y_n < \alpha < 0 \ for \ n \geq N$ .Taking summation from N to n-1 in (5) ,we obtain ,

$$\sum_{i=N}^{n-1} \Delta w_i \geq -\sum_{i=N}^{n-1} q_i G\left(y_{\sigma(i)}\right) > -G(\alpha)\sum_{i=N}^{n-1} q_i$$

$$\Rightarrow \lim_{n\to\infty}\left(w_n - w_N\right) > \lim_{n\to\infty} -G(\alpha)\sum_{i=N}^{n-1} q_i > \infty$$

$$\Rightarrow \lim_{n\to\infty} w_n > \infty \Rightarrow l = \infty$$

Which is a contradiction .Hence our claim holds.So $\lim_{n\to\infty} z_n = 0$

Consequently $\lim_{n\to\infty} w_n = 0$ .Hence $w_n < 0$,because it is increasing .This completely the proof.

Theorem :2

Suppose that $f_n \geq 0$ .Let $p_n = p \neq 1$ .Suppose (C0-C3) holds.If y_n be an eventually negative solution of (2.1) .Set w_n as in (3).Then the following statements hold.

(a) w_n is an increasing sequence and either

(7) $\lim_{n\to\infty} w_n = \infty$ or,

(8) $\lim_{n\to\infty} w_n = 0$

(b) The following statements are equivalent

(i) $\lim_{n\to\infty} w_n = \infty$

(ii) $p_n > 1$

(iii) $\lim_{n\to\infty} y_n = -\infty$

(C) The following statements are equivalent.

(i) $\lim_{n\to\infty} w_n = 0$

(ii) $p < 1$

(iii) $\lim_{n\to\infty} y_n = 0$

Proof:

(a)

From (1) and (3) ,we obtain (5) .Which clearly indicates w_n is monotonic increasing Sequence.Hence, $\lim_{n\to\infty} w_n = l$ or $\lim_{n\to\infty} w_n = \infty$. If l is finite then $\lim_{n\to\infty} z_n = l$ .Next we prove that $\limsup_{n\to\infty} y_n = 0$ .As in the proof of previous Theorem (2) ,then by lemma (4) we get $\lim_{n\to\infty} z_n = 0$ $\Rightarrow l = 0$. So proof of part(a) is complete.

(b)

Let (i) holds good. That is $\lim_{n\to\infty} w_n = \infty$ .By the use of (C2) we obtain $\lim_{n\to\infty} z_n = \infty$ .That is $\lim_{n\to\infty}\left(y_n - p_n y_{\tau(n)}\right) = \infty$. Note that $p_n = p$ .Then clearly it follows that p must be negative and y_n must be bounded. Therefore by lemma (2) , there exist $n^* \ge n_0$ such that $y_n^* = \min\{y_k, k \le n^*\}$ .Then

$$0 < z_n^* = y_{n^*} - p_n y_{\tau(n^*)} < y_{n^*} - p_n y_{n^*} = (1-p) y_{n^*}$$

.Since ,

$$p - 1 > 0 \Rightarrow p > 1$$

Therefore (i) implies (ii)

Let (ii) holds that is $p > 1$ .By (a) either (7) holds or (8) holds .If (8) holds then we conclude that $w_n < 0$ as w_n is increasing.Since $F_n > 0$, therefore by (3) we get $z_n < 0$ for large n.

Thus $y_n < p_n y_{\tau(n)} \le y_{\tau(n)} \Rightarrow \limsup_{n\to\infty} y_n < 0$,by lemma (1.5.1).But as at (6) we can show $\limsup_{n\to\infty} y_n = 0$,which is a contradiction. Hence we are left with a possibility of (7) only .From this and (C2) ,it follows that $\lim_{n\to\infty} z_n = \infty$. Further we have $z_n < -p y_{\tau(n)}$ .Hence $y_{\tau(n)} < \frac{z_n}{-p} \Rightarrow \limsup_{n\to\infty} y_n > \lim_{n\to\infty} \frac{z_n}{-p} = -\infty$,so $\lim_{n\to\infty} y_n = -\infty$, and (iii) is proved.

Next to prove (iii) implies (i).

Let $\lim_{n\to\infty} y_n = -\infty$ .There fore for some $m<0$, there exist N such that $n \ge n_1$, $y_n < m < 0$.

$$\Rightarrow \sum_{n=N}^{\infty} q_n G\left(y_{\sigma(n)}\right) < \sum_{n=N}^{\infty} q_n G(m) \to -\infty \;.$$

But if w_n is bounded then $\sum_{n=N}^{\infty} q_n G\left(y_{\sigma(n)}\right) > -\infty$ .Which is a contradiction. Hence w_n is unbounded. Therefore $\lim_{n\to\infty} w_n = -\infty$ .So by part (a) ,part(b) proof is over.

(c)

Let (i) hold, that is $\lim_{n\to\infty} w_n = 0$ .From the fact that w_n is monotonic increasing ,so we get $w_n < 0$,since $f_n \le 0$ .Therefore from (3) we get $z_n < w_n$,hence $z_n < 0$ .Assume if possible $p \ge 1 \Rightarrow y_n < p_n y_{\tau(n)} \le y_{\tau(n)}$,so by lemma (1) , $\Rightarrow y_n < m$ for some $m < 0$. Hence, $\sum_{n=N}^{\infty} q_n G\left(y_{\sigma(n)}\right) \to \infty$ by (C3) But as $\lim_{n\to\infty} w_n = 0$ by taking summation of (5) $\sum_{n=N}^{\infty} q_n G\left(y_{\sigma(n)}\right) > -\infty$. Which is a contradiction .So $p \ge 1$ is impossible. Thus (ii) holds good.

Next to show that (ii) $\Rightarrow$ (iii).

Let (ii) holds That is $p < 1$.Then two cases arises , $p \le 0$ or $p \in (0,1)$.Let $p \le 0$ .Now we claim that y_n is bounded sequence. Otherwise w_n is unbounded. By part (a) of this lemma ,we have (7) holds. Again by part(b) of this lemma , $p > 1$.Which is a contradiction. Hence our claim holds and $\limsup_{n\to\infty} y_n$ and $\liminf_{n\to\infty} y_n$ exists. By part (a) it is clear that (8) holds. Thus $\lim_{n\to\infty} z_n = 0$ by remark (2). Applying lemma (1) ,we get $\lim_{n\to\infty} y_n = 0$.

Next consider $p \in (0,1)$,we claim that (8) holds.Otherwsie by part(a) wee that (7) holds.So $\lim_{n\to\infty} z_n = \infty \Rightarrow z_n > 0$ for large n. Hence $y_n - p_n y_{\tau(n)} > 0 \Rightarrow y_n > p_n y_{\tau(n)} \ge y_{\tau(n)}$ (since y_n is negative).So y_n is bounded by lemma (1). This implies w_n is bounded ,which is a contradiction. Hence (2.228) holds .This implies that $\lim_{n\to\infty} z_n = 0$.Then we claim that y_n is bounded ,Otherwise by lemma (2) ,we find a sequence $\{n_m\}$ such that $m \to \infty \Rightarrow \{n_m\} \to \infty$ and $\{y_{n_m}\} \to \infty$ and $\{y_{n_m}\} = \min\{y_n : n_1 \le n \le n_m\}$ .Note that by remark (2.12) ,there exist some $\alpha > 0$ such that $|F_{n_m}| < \alpha \in R.$ Hence $w_{n_m} = y_{n_m} - p y_{\tau(n_m)} + F_{n_m} < (1-p) y_{n_m} + \alpha \to -\infty$ as $n \to \infty$ .Which is a contradiction .So y_n is bounded.This implies $\limsup_{n\to\infty} y_n$ and $\liminf_{n\to\infty} y_n$ exists .Then

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applying Lemma (3) we obtain $\limsup_{n\to\infty} y_n = 0$ Thus $\lim_{n\to\infty} y_n = 0$.

Hence (ii) $\Rightarrow$ (iii).

Next to show (iii) $\Rightarrow$ (i).

Let $\lim_{n\to\infty} y_n = 0$ .To show that $\lim_{n\to\infty} w_n = 0$ .Clearly y_n is bounded.This implies w_n is bounded. So by part(a) $\lim_{n\to\infty} w_n = 0$,because (7) cannot hold. So the proof of (i) is over.

Lemma:4

Let (C0),(C1) and (C3) holds good and $p \neq -1$.Then $p_n = p < 1$ iff every nonoscillatory solutions of the NDDE(2*) tends to zero as $n \to \infty$

Lemma:5

Suppose that (C0) ,(C2) and (C3) holds good.

(i) If p_n satisfies one of the conditions (D1),(D2) or (D3) then every solution of (1) oscillates or tends to zero as $n \to \infty$

(ii) If p_n satisfies (D4) then every bounded solution of (1) oscillates or tends to zero as $n \to \infty$

3 ILLUSTRATIVE EXAMPLES

In this section, two examples are given to illustrate our theorems.

Example :- 1

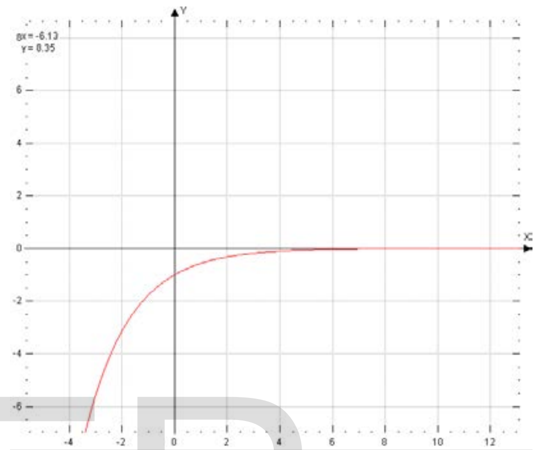
(9) $$\Delta\left(y_n - \frac{1}{2\pi} y_{n-2}\right) + \frac{2\sqrt{\pi}-1}{2\pi^{\alpha/2}} y_{n-\alpha} = 0 \ ,\alpha > 0$$

Here $p = \frac{1}{2\pi} \in (0,1)$, $\tau(n) = n-2$, $\sigma(n) = n-\alpha$

$q = \frac{2\sqrt{\pi}-1}{2\pi^{\alpha/2}}$, *which satisfies* $(C3)$, $f_n = 0$

Here $z_n = w_n = -\frac{1}{2}\pi^{-n/2} < 0$ *and* $w_n \to 0$ *as* $n \to \infty$

Clearly $y_n = -\pi^{-n/2}$ *is a solution of* (9)



This example illustrates Theorem (1 part(a) implies part (c)),Theorem(2 part (a) implies part(c)),Lemma (4),(5 (i)).

Example:- 2

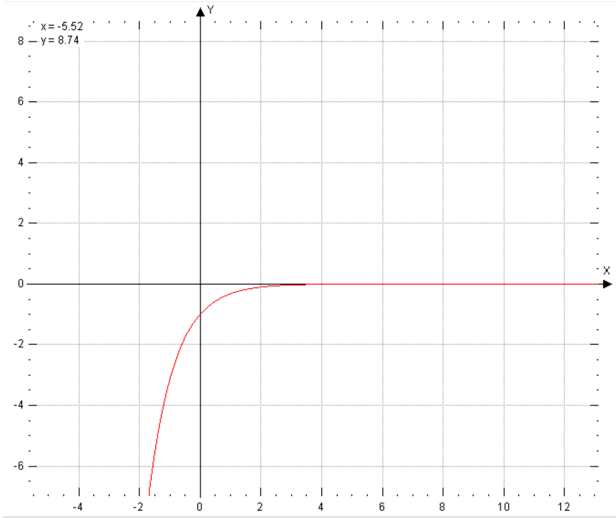
(10) $$\Delta\left(y_n - 2y_{n-\alpha}\right) + \left(\pi^b - 1\right) y_{n-\beta} = 0 \ ,\alpha > 0, \beta > 0$$

Here $p = 2\pi^\alpha > 1$, $\tau(n) = n-\alpha$, $\sigma(n) = n-\beta$

$q = \pi^b - 1$, *which satisfies* $(C3)$, $f_n = 0$

Here $z_n = w_n = \pi^n > 0$ *and* $w_n \to 0$ *as* $n \to \infty$

Clearly $y_n = -\pi^{-n}$ *is a solution of* (10)



This example illustrates Theorem (1 part(a) implies part (b),Theorem (2 part (a) implies part(b) ,Lemma(4) ,(5 (ii)).

None of the known outcomes can be connected to the above cases.

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